1 A translation of K_3 into Modal Logic

In [1] a translation from any formula A of K_3 to a formula $A^\Box$ of modal logic is specified as follows:

- $\neg$ stands for K_3 negation and $\sim$ stands for classical negation
- $(p)^\Box := \Box p$
- $(\neg p)^\Box := \Box \sim p$
- $(A \wedge B)^\Box := A^\Box \wedge B^\Box$
- $(A \vee B)^\Box := A^\Box \vee B^\Box$ [1]

(Note: no translation for the negation of arbitrary formulas is given, but we can just convert K_3 formulas into negation normal form.)

The sequent $A \vdash B$ is provable in K_3 iff the translated formula $A^\Box \supset B^\Box$ is a consequence of the 'Deontic' axiom **D**:

$$\Box p \supset \Diamond p$$

where only *non-modal* inference steps are used.

> The reason why this holds is that the proofs in K_3 essentially carry over into modal logic, and the fundamental $\sim (p \wedge \neg p)$ becomes $\sim (\Box p \wedge \Box \sim p)$, easily seen to be equivalent to **D**. After the transformation $\Box p$ and $\Box \sim p$ still behave like distinct variables.
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> This translation is very simple, but it is interesting to see how various three-valued inferences translate into modal formulas. For example, the sequent $p \vdash q \vee \neg q$ becomes $\Box p \supset (\Box q \vee \Box \sim q)$, which is not provable from **D**. On the other hand, the correct $\neg p \vdash \sim p$ becomes $\Box \sim p \supset \sim \Box p$, essentially the same as **D**.
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> It is not necessary to be very specific about the target modal language. [1, p. 73.]

I take this to mean that the normal modal logic K + **D** will do.

2 A translation of LP into Modal Logic

The following seems to work. We have the same translation:

- $\neg$ stands for LP negation and $\sim$ stands for classical negation
- $(p)^\Box := \Box p$
- $(\neg p)^\Box := \Box \sim p$

[1] There is a typo in the text, which has $(A \vee B)^\Box := A^\Box \vee B$

- $(A \wedge B)^{\Box} := A^{\Box} \wedge B^{\Box}$
- $(A \vee B)^{\Box} := A^{\Box} \vee B^{\Box}$

The sequent $A \vdash B$ is provable in LP iff the translated formula $A^{\Box} \supset B^{\Box}$ is provable in the normal modal logic K + **CD**, where **CD** is the 'Uniqueness' axiom:

$$\Diamond A \supset \Box A$$

Here are some examples:

- $p \wedge \neg p \nvdash_{LP} q$
  - $p^{\Box} \wedge \neg p^{\Box} \vdash q^{\Box}$
  - $\Box p \wedge \Box \sim p \nvdash \Box q$
- $q \vdash_{LP} p \vee \neg p$
  - $q^{\Box} \vdash p^{\Box} \vee \neg p^{\Box}$
  - $\Box q \vdash \Box p \vee \Box \sim p$
- $p \wedge (\neg p \vee q) \nvdash_{LP} q$
  - $p^{\Box} \wedge (\neg p \vee q)^{\Box} \nvdash q^{\Box}$
  - $\Box p \wedge (\Box \sim p \vee \Box q) \nvdash \Box q$

References

[1] Busch, Douglas, 'Sequent Formalizations of Three-Valued Logic'. In Patrick Doherty (ed.) *Partiality, Modality and Nonmonotonicity.* CSLI Publications, 1996, pp. 45-75.