Truthlikeness and the Lottery Paradox via the Preface Paradox

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Abstract

In a recent paper Cevolani and Schurz (C&S) propose a novel solution to the Preface Paradox that appeals to the notion of expected truthlikeness. This discussion note extends and analyses their approach by applying it to the related Lottery Paradox.

Keywords: Truthlikeness, Lottery Paradox, Preface Paradox

1 The Preface Paradox and Truthlikeness

In a recent paper Cevolani and Schurz (C&S) [forthcoming] propose a novel solution to the Preface Paradox. It appeals to the notion of *expected truthlikeness*, which involves a combination of truthlikeness and probability. With a literal ($\pm p_i$) being either an atomic statement p_i or its negation $\neg p_i$, the basic account of truthlikeness employed deals exclusively with propositional statements that are conjunctions of literals; for example, $p_1 \wedge \neg p_2$. Where n stands for the number of atoms in a given logical space, statements of the form $\pm p_1 \wedge \pm p_2 \wedge ... \wedge \pm p_n$ are termed *constituents* of the language or *state descriptions*, since they completely describe a state/model.

The truthlikeness of statement h is given by the following measure:

$$\mathrm{Tr}_{\varphi}(h) = \tfrac{t}{n} - \varphi\tfrac{f}{n}$$

where $\frac{t}{n}$ is the normalized number of true claims t and $\frac{f}{n}$ is the normalized number of false claims f, weighted by a parameter $\varphi > 0$. $\varphi = 1$ indicates that correctness and error are weighed equally, $\varphi > 1$ that error outweighs correctness and $\varphi < 1$ that correctness outweighs error.

The expected truthlikeness of statement h given evidence e is defined as follows:

$$\mathrm{ETr}_{\varphi}(h|e) = \Sigma_{w_i}\mathrm{P}(w_i|e)\mathrm{Tr}_{\varphi}(h, w_i)$$

where w_i stands for state i, $\mathrm{P}(w_i|e)$ its probability given e and $\mathrm{Tr}_{\varphi}(h, w_i)$ stands for the truthlikeness of h given that w_i is the true state. As C&S state, given this best estimation of the truthlikeness of h based on evidence e, an inquirer could adopt the following strategy of rational acceptance: accept that statement h which maximises expected truthlikeness.

Take the following brief description of the Preface Paradox as outlined by C&S: Adam, an academic historian, publishes a big work containing a great number m of claims $b_1, ..., b_m$. Although he is ready to claim that each b_i and hence their conjunction b is true, he is also aware of his own fallibility and thus acknowledges that his book is bound to contain some error. Adam is thus seemingly in a position where he is entitled to accept both b and $\neg b$ and paradox ensues.

According to C&S's solution,

> what Adam asserts by publishing the book is that b is his best attempt to approximate the truth about the domain under inquiry - in other words, that b maximizes expected truthlikeness, given his assessment of the relevant probabilities and the available evidence e. Still, as Adam makes clear in the preface of his book, b may be likely false, or even already falsified by e. [Cevolani and Schurz forthcoming: 10]

They show that by accepting the conjunction of all sufficiently probable atomic claims about the domain, an inquirer has a simple way to maximise expected truthlikeness. This is formally captured in their Theorem 2:

Theorem 2. If b is the conjunction of all and only basic statements $b_1, ..., b_m$ such that

$$P(b_i|e) > \sigma = \varphi/(\varphi + 1), \text{ then } ETr_\varphi(b|e) \text{ is maximal}$$

Thus Adam can rationally accept b as the statement with the highest expected truthlikeness given e despite his acknowledgement in the preface that b's plain truth is unlikely. As we shall now see, applying this truthlikeness approach to the Lottery Paradox actually provides a starker concrete example of a scenario where the best choice is a statement which necessarily has a probability of zero given the evidence.

2 The Lottery Paradox and Truthlikeness

Kyburg's well-known lottery paradox [Kyburg 1961] arises from the following three principles:

1. It is rational to accept a statement that has a significantly high probability.
2. If it is rational to accept a statement A and it is rational to accept another statement B, then it is rational to accept their conjunction $A \wedge B$.
3. It is not rational to accept a contradictory statement.

Consider a fair 1000-ticket lottery that we know has exactly one winning ticket. It follows that for any ticket i in this lottery the probability of it losing is 0.999. A probability greater than or equal to 0.999 is very high and given the first principle then for any individual ticket i it is rational to accept that it will lose. However, this and the second principle of conjunction closure would entail that it is rational to accept the statement that every ticket will lose, which contradicts the original piece of knowledge that some ticket will win.

Proposed treatments of this paradox abound. Kyburg himself addresses the issue by accepting 1 and rejecting 2. On the other hand, there are many who accept 2 and reject 1. An alternative approach is to appeal to the epistemic goal of maximising truth and minimising falsity. Douven [2008] suggests something along these lines:

> Given the same 10-ticket lottery, accept of nine tickets that they will lose, and believe of the remaining one that it will win. Clearly, there is no longer an inconsistency in your beliefs about the lottery. And there is a 90% chance that you have added eight true beliefs and two false ones to your stock of beliefs - still not a bad score (or if you think it is, take a 100-ticket lottery, or ...). Better yet, there is even a 10% chance that you have added nothing but true beliefs. [Douven 2008: 7]

The idea is that one could arbitrarily make a selection of 9 winners and 1 loser and bypass the need to reject Principle 1 or Principle 2, with the result being a consistent statement that accords with the goal of truth.

If however, as the following example demonstrates, our goal is to maximise expected truthlikeness, we must forsake consistency. To simplify matters, suppose that we have a 10-ticket lottery situation with the following settings:

- l_i stands for ticket i will lose
- $\varphi = 1$, $\sigma = 0.5$
- $e = (\neg l_1 \wedge l_2 \wedge ... \wedge l_{10}) \vee ... \vee (l_1 \wedge l_2 \wedge ... \wedge \neg l_{10})$
- $\mathrm{P}(l_i|e) = 0.9$

In this case, it is actually the statement $l = l_1 \wedge l_2 \wedge l_3 \wedge l_4 \wedge l_5 \wedge l_6 \wedge l_7 \wedge l_8 \wedge l_9 \wedge l_{10}$ that has the highest expected truthlikeness: $\mathrm{ETr}_{\varphi}(l|e) = 10 \times 0.1 \times (\frac{9}{10} - \frac{1}{10}) = 0.8$, even though

$P(l|e) = 0$. In terms of Theorem 2, for each l_i, $P(l_i|e) = 0.9 > 0.5$; φ has to reach a value of 9 before falsity sufficiently outweighs truth. On the other hand, take a state description consistent with the evidence such as $c = \neg l_1 \wedge l_2 \wedge l_3 \wedge l_4 \wedge l_5 \wedge l_6 \wedge l_7 \wedge l_8 \wedge l_9 \wedge l_{10}$ in line with Douven's suggestion: $\text{ETr}_{\varphi}(c|e) = (9 \times 0.1 \times (\frac{8}{10} - \frac{2}{10})) + (1 \times 0.1 \times (\frac{10}{10})) = 0.64$.

So what can we take from all of this? Although we know that $\neg l$, we are in some sense entitled to adopt l because it has the highest expected truthlikeness. Of course, this does not mean accepting that l is true. Rather, it suggests some sort of utilitarian commitment. For example, if one were to bet on a maximal statement and receive an equal amount for each ticket they got right, then they would be wise to bet on l. In this sense, it is rational to adopt a contradictory statement.

One issue that might be had with this approach is that if belief/adoption is closed under logical consequence, then all statements, both false and true, will automatically be added due to the classical principle of explosion. This would actually result in lower truthlikeness, as Douven similarly notes:

> So why not adopt all these beliefs? One answer is that if belief is closed under logical consequence, then by believing of each ticket that it will lose you will automatically add all propositions - true and false ones - to your stock of beliefs (for believing of all the tickets that they will lose contradicts your belief that the lottery has a winner). The result is definitely not a body of beliefs with a favorable truth-falsity ratio. [Douven 2008: 7]

But it is reasonable to adopt a paraconsistent approach here and employ a doxastic consequence relation such that not everything follows from $l \wedge \neg l$ [Priest, Tanaka and Weber 2016]. In fact, a paraconsistent treatment of truthlikeness is possible (for those readers who are interested a demonstration can be found in the appendix).

Before ending, there is one more point to consider. Rather than devising a solution to the lottery paradox by focused attempts to reject or modify the principals central to its original formulation, the framework outlined here offers an alternative approach, where the goal is not to believe what is probably true but to adopt what is by estimation maximally truthlike. In terms of the original formulation, we saw that one option is to reject the sufficiency of high probability. Another is to reject conjunction closure, because its application eventually leads to contradiction. Still, suppose that the first principle was strengthened as follows: it is rational to accept a statement if and only if it has a significantly high probability. This would imply a failure of conjunction closure as its application would generally lead to a conjunction which falls below an acceptable threshold. For example, suppose that we impose the modest requirement that it is rational to accept a proposition p if and only if $\mathrm{P}(p) > 0.5$. In the case of a 10-ticket lottery, probabilities start to fall below this threshold after six conjoined atoms:

- $\mathrm{P}(l_1 \wedge l_2 \wedge l_3 | e) = 0.7$
- $\mathrm{P}(l_4 \wedge l_5 \wedge l_6 | e) = 0.7$
- $\mathrm{P}(l_1 \wedge l_2 \wedge l_3 \wedge l_4 \wedge l_5 \wedge l_6 | e) = 0.4$

In spite of this, a truthlikeness-based approach to acceptability permits definitions whereby conjunction closure does not lead to such threshold failures. For a simple example, when $\varphi = 1$, the highest expected truthlikeness value for one atom is $\frac{1}{n}$ and the lowest is $-\frac{1}{n}$.[1] Suppose there is a threshold τ, such that $0 < \tau \leq \frac{1}{n}$. With these parameters, we can establish the following simple principles:

[1]When an atom p is made true by a state w, $\mathrm{Tr}(p, w) = \frac{1}{n}$. When calculating ETr, if for all w_i that add non-zero to $\Sigma_{w_i}\mathrm{P}(w_i)$ it is the case that $\mathrm{Tr}(p, w_i) = \frac{1}{n}$, then since $\Sigma_{w_i}\mathrm{P}(w_i) = 1$, the maximum is $\Sigma_{w_i}\mathrm{P}(w_i)\frac{1}{n} = \frac{1}{n}$. The converse applies for the lowest of $-\frac{1}{n}$.

1. A conjunction of $m \geq 1$ atoms is adoptable if and only if its expected truthlikeness x is such that $x \geq \tau m$
2. If statements A and B share no atoms and are adoptable, then $A \wedge B$ is adoptable

In this case, we can aggregate statements without conflict between principles 1 and 2 and truthlikeness increases. The final result of such aggregation might be a statement that is inconsistent with the evidence, but that is the price to pay for truthlikeness maximisation.

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Appendix

The Logic of Paradox (*LP*) [Priest 1979: Priest, Tanaka and Weber 2016] extends classical logic with another designated truth value B (both truth and false). To simplify matters for this explication, consider a 3-ticket lottery (l_1, l_2 and l_3) with the same type of paradox scenario. This setup generates 27 possible states in *LP*, as tabulated in Table 1.

State	l_1	l_2	l_3	State	l_1	l_2	l_3	State	l_1	l_2	l_3
w_1	T	T	T	w_{10}	B	T	T	w_{19}	F	T	T
w_2	T	T	B	w_{11}	B	T	B	w_{20}	F	T	B
w_3	T	T	F	w_{12}	B	T	F	w_{21}	F	T	F
w_4	T	B	T	w_{13}	B	B	T	w_{22}	F	B	T
w_5	T	B	B	w_{14}	B	B	B	w_{23}	F	B	B
w_6	T	B	F	w_{15}	B	B	F	w_{24}	F	B	F
w_7	T	F	T	w_{16}	B	F	T	w_{25}	F	F	T
w_8	T	F	B	w_{17}	B	F	B	w_{26}	F	F	B
w_9	T	F	F	w_{18}	B	F	F	w_{27}	F	F	F

Table 1: *LP* Truth Table for 3-Proposition Logical Space

In this system, a contradictory (classically unsatisfiable) statement such as $l_1 \wedge \neg l_1 \wedge l_2 \wedge \neg l_3$ is no longer unsatisfiable. Rather, this statement would be true in the states w_{11}, w_{12}, w_{14} and w_{15}; a proposition p is satisfied in a state (model) when its valuation is T or B, $\neg p$ is satisfied when p is F or B and $p \wedge \neg p$ is only satisfied when p is B. In this scenario, the evidence proposition e and the proposition l that all tickets will lose simply translate to:

- $e = (\neg l_1 \wedge l_2 \wedge l_3) \vee (l_1 \wedge \neg l_2 \wedge l_3) \vee (l_1 \wedge l_2 \wedge \neg l_3)$
- $l = l_1 \wedge l_2 \wedge l_3$

Whilst their conjunction is unsatisfiable in classical logic, in *LP* $e \wedge l$ is satisfied in w_2, w_4, w_5, w_{10}, w_{11}, w_{13} and w_{14}. This set of states corresponds to the statement $(l_1 \wedge \neg l_1 \wedge l_2 \wedge l_3) \vee (l_1 \wedge l_2 \wedge \neg l_2 \wedge l_3) \vee (l_1 \wedge l_2 \wedge l_3 \wedge \neg l_3)$. As will now be exemplified, we can use this paraconsistent logic as an inconsistency tolerant tool to reason and calculate.

D'Alfonso [2011] extends the Tichy-Oddie (TO) average distance likeness approach [Oddie 2016] to get a truthlikeness measure over *LP*.[2] Unlike the C&S measure, which as we have seen only deals with certain conjunctive statements, the TO measure applies to any propositional statement in a given logical space. Nonetheless, both measures share a property known as c-monotonicity [Cevolani, Crupi and Festa 2011] and when restricted to the set of such conjunctive statements provide ordinally equivalent results.[3] At any rate, this type of paraconsistent extension can be applied in a couple of ways to C&S's measure, which we recall is:

$$\mathrm{Tr}_\varphi(h) = \frac{t}{n} - \varphi\frac{f}{n}$$

[2] The TO truthlikeness measure of a statement A against the true state w is $\mathrm{Tr}(A, w) = 1 - \Delta(A, w)$, where $\Delta(A, w)$ is the average difference of atom valuations between w and the models of A.

[3] C-monotonicity is a property whereby given two conjunctive statements A and B, if B has more true atoms and less false atoms than A, then the truthlikeness of B is greater than the truthlikeness of A.

With the first way, when assessing the values for t and f against a given state w, the classical conditions remain. Against a classical truth value of T or F, a contradiction $p \wedge \neg p$ simply cancels itself out, since one of the conjuncts adds to t and the other adds to f. This simple approach to extension adds the following for B: if a literal (p or $\neg p$) is in the conjunctive statement h and p has the value B in w, then do not add anything to t or f.

For example, suppose that the true state is w_3 and $\varphi = 1$. Then:

- $\mathrm{Tr}(l_1 \wedge l_2 \wedge \neg l_3) = 1$
- $\mathrm{Tr}(l_1 \wedge \neg l_1) = 0$
- $\mathrm{Tr}(l_1 \wedge \neg l_1 \wedge l_2) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

If however we were to measure statements against w_{10} then:

- $\mathrm{Tr}(l_1 \wedge l_2 \wedge \neg l_3) = 0$
- $\mathrm{Tr}(l_1 \wedge \neg l_1) = 0$
- $\mathrm{Tr}(l_1 \wedge \neg l_1 \wedge l_2) = \frac{1}{3}$

In this paraconsistent setting the evidence formula $e = (\neg l_1 \wedge l_2 \wedge l_3) \vee (l_1 \wedge \neg l_2 \wedge l_3) \vee (l_1 \wedge l_2 \wedge \neg l_3)$ is satisfied in the following set of states:

$$\{w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_{10}, w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17}, w_{19}, w_{20}, w_{22}, w_{23}\}.$$

When a standard uniform probability of $\frac{1}{19}$ is assigned to these 19 states and these figures are plugged into the estimated truthlikeness formula then the statement with highest estimated truthlikeness, as in the classical case, is $l_1 \wedge l_2 \wedge l_3$, with a measure of 0.16. A state description such as $\neg l_1 \wedge l_2 \wedge l_3$, which asserts that there will be a single winner, receives a lower measure of 0.05. Thus in accordance with our main result in the classical case, it is the statement which asserts that each ticket will lose which has maximum estimated truthlikeness in this lottery scenario.

This approach in effect ignores the B value and is sufficient for our instrumental purposes. It is worth mentioning that another approach (which could be used by dialetheic frameworks that endorse the actuality of paraconsistent states) to extending C&S's measure could be to add the following for B instead: if $p \wedge \neg p$ is in the conjunctive statement h and p has the value B in w, then add 1 to t (when p has value B but only one of p or $\neg p$ appear in h, add nothing).

This would mean that $p \wedge \neg p$ fully counts towards Tr when p has the value B in w and that the statement $d = l_1 \wedge \neg l_1 \wedge l_2 \wedge \neg l_2 \wedge l_3 \wedge \neg l_3$ has a maximum Tr of 1 given w_{14} as the actual state. In fact, given the evidence statement e as defined above it is actually d which would get the highest estimated truthlikeness value using this paraconsistent measure, although $l = l_1 \wedge l_2 \wedge l_3$ would still be the highest ranked classically consistent statement.

Obviously in the lottery scenario the statement d would be strange as it expresses the case that all tickets will both lose and win. Barring a selection strategy where the chosen statement is the highest *classical* statement, the key here is to use an evidence statement that more accurately/strictly represents the evidence to get a result where l comes out on top.

In the language of LP, literals are no longer sufficient to express state descriptions, as p corresponds to T or B and $\neg p$ to F or B. For example, in classical logic the statement $l_1 \wedge l_2 \wedge l_3$ would uniquely correspond to the state w_1. But in LP, it could be satisfied by not only w_1, but also by w_2, w_4, w_5, w_{10}, w_{11}, w_{13} and w_{14}. Thus let us use $X(l_i)$ to mean that l_i will have exactly the truth value X; so $T(l_i)$ means that ticket i will only lose, $F(l_i)$ that it will only win and $B(l_i)$ that it will both win and lose. In terms of this 3-ticket lottery, our evidence that exactly one ticket will win and only win would be

$e = (F(l_1) \wedge T(l_2) \wedge T(l_3)) \vee (T(l_1) \wedge F(l_2) \wedge T(l_3)) \vee (T(l_1) \wedge T(l_2) \wedge F(l_3))$, which corresponds to the set $\{w_3, w_7, w_{19}\}$.

Given this specific statement, $\mathrm{ETr}_\varphi(l|e)$ would again turn out to be maximal, once again in accordance with our main result that it is the statement which asserts that each ticket will only lose which has maximum estimated truthlikeness in the lottery scenario. This second of the two modified versions of C&S's measure gives the same results as the paraconsistent version of TO's measure mentioned above, which would rank $\mathrm{ETr}_\varphi(l|e)$ as maximal amongst all statements in the logical space, not just conjunctive ones.