# Integrity Constraints and Truthlikeness in Belief Merging

## 1 Belief merging with the aim of truthlikeness

It has been established in [1] that methods of belief merging suitable for the aim of truthlikeness do not necessarily conform to standard belief merging frameworks [2]. Following on from this work, in this note I consider how integrity constraints would work for belief merging in a truthlikeness context. The two operators featured in this discussion are $\Delta^{\min,\Sigma}$ and $\Delta^{\text{avg},\Sigma}$ as detailed in [1]. Table 1 depicts the interpretations used.

| | $p_1$ | $p_2$ | $p_3$ |
|---|---|---|---|
| $w_1$ | 1 | 1 | 1 |
| $w_2$ | 1 | 1 | 0 |
| $w_3$ | 1 | 0 | 1 |
| $w_4$ | 1 | 0 | 0 |
| $w_5$ | 0 | 1 | 1 |
| $w_6$ | 0 | 1 | 0 |
| $w_7$ | 0 | 0 | 1 |
| $w_8$ | 0 | 0 | 0 |

Table 1: The eight interpretations/possible worlds in the logical space based on the three atomic formulas $p_1$, $p_2$ and $p_3$.

### 1.1 Integrity constraints in the merging of opinions/preferences

In the context of merging opinions/preferences, an integrity constraint indicates a set of states that cannot feature in the outcome. This is not because such states are deemed not possible, but because for one reason or another they are not acceptable. Despite the final unacceptability of such states, they nonetheless can feature in the belief bases that are to be aggregated. As the following example shows, despite being excluded from any final result, a state that features in the integrity constraints can nonetheless influence the final result.

Suppose we have the following details:

- $K_1 = \{w_1, w_2\}$
- $K_2 = \{w_2, w_4\}$
- $K_3 = \{w_5, w_6\}$
- $E = \{K_1, K_2, K_3\}$

- $IC = \neg(p_1 \wedge p_2 \wedge \neg p_3)[\neg w_2]$

Table 2 contains a tabulation of the minimal distance between each interpretation and each belief base and the sum aggregation function result, with the results in bold being those that are selected.

| State | $K_1$ | $K_2$ | $K_3$ | Sum |
|---|---|---|---|---|
| $w_1$ | 1 | 0 | 1 | **2** |
| $w_2$ | 0 | 0 | 1 | 1 |
| $w_3$ | 1 | 1 | 2 | 4 |
| $w_4$ | 0 | 1 | 2 | 3 |
| $w_5$ | 2 | 1 | 0 | 3 |
| $w_6$ | 1 | 1 | 0 | **2** |
| $w_7$ | 2 | 2 | 1 | 5 |
| $w_8$ | 1 | 2 | 1 | 4 |

Table 2: Tabulated distances

Although $w_2$ is at a minimum distance, since $IC = \neg w_2$, the next closest states are chosen: $\Delta^{\min,\Sigma}(E) = \{w_1, w_6\}$. Despite $w_2$ being excluded, it still has an influence on the final result, for if it were removed from the input belief sets the result would be different, as the following shows:

- $K_1 = \{w_1\}$
- $K_2 = \{w_4\}$
- $K_3 = \{w_5, w_6\}$
- $E = \{K_1, K_2, K_3\}$
- $IC = \neg w_2$

  Table 3 contains a tabulation for this modified setup.

| State | $K_1$ | $K_2$ | $K_3$ | Sum |
|---|---|---|---|---|
| $w_1$ | 0 | 2 | 1 | **3** |
| $w_2$ | 1 | 1 | 1 | **3** |
| $w_3$ | 1 | 1 | 2 | 4 |
| $w_4$ | 2 | 0 | 2 | 4 |
| $w_5$ | 1 | 3 | 0 | 4 |
| $w_6$ | 2 | 2 | 0 | 4 |
| $w_7$ | 2 | 2 | 1 | 5 |
| $w_8$ | 3 | 1 | 1 | 5 |

Table 3: Tabulated distances

Obviously the result differs: $\Delta^{\min,\Sigma}(E) = \{w_1, w_2\}$.

## 1.2 Integrity constraints in the merging of factual data

As we have seen, the standard belief merging framework incorporates integrity constraints by selecting the states with minimal distance that also satisfy the integrity constraints. In the context of merging factual data, an integrity constraint indicates a set of states that cannot feature in the final outcome. Unlike in the case of merging opinions/preferences however, such states are deemed not possible. Thus as well as being excluded from the final result, it makes sense that such states should also be removed from the input at some stage; since the presence of such states can have an influence on the final result, if a state is judged to be impossible by the merger, then it should be removed prior to the merging process.

Suppose we have the following two belief bases:

- $K_1 = \{w_1, w_2\}$
- $K_2 = \{w_3, w_4, w_5\}$
- $\Delta^{\text{avg},\Sigma}(\{K_1, K_2\}) = \{w_1\}$

Now suppose that $IC = \neg w_5$ is the integrity constraint. If this integrity constraint were to only be enforced in the final result, then since $\Delta^{\text{avg},\Sigma}(\{K_1, K_2\})$ satisfies $\neg w_5$, $\{w_1\}$ would be the final result. If however $\neg w_5$ is enforced earlier so that the integrity constraint is imposed on the input belief bases, then the result differs. This approach involves 'preparing' each base by firstly conditionalising on the integrity constraint. Following the procedure established in [1], we convert $\Delta^{\text{avg},\Sigma}$ to a probability aggregation plus maximised estimated truthlikeness procedure:

- $K_1 = \{w_1, w_2\}$
- $K_2 = \{w_3, w_4, w_5\}$
- $Pr_{K_1}(w_1) = Pr_{K_1}(w_2) = \frac{1}{2}$
- $Pr_{K_2}(w_3) = Pr_{K_2}(w_4) = Pr_{K_2}(w_5) = \frac{1}{3}$
- $IC = \neg w_5$
- Where $K_{2'}$ stands for the new probability distribution given $\neg w_5$, we get $Pr_{K_2}(w_3|\neg w_5) = Pr_{K_{2'}}(w_3) = Pr_{K_{2'}}(w_4) = \frac{1}{2}$
- Given this combined probability distribution, the final result is $\Delta^{\text{avg},\Sigma}(\{K_1, K_{2'}\}) = \{w_1, w_2, w_3, w_4\}$

Alternatively, we could first aggregate all the input probabilities then conditionalise against the integrity constraint before estimating maximum truthlikeness?

Next we asses two postulates given by Konieczny and Pérez [2] in supplementing their merging framework with integrity constraints.

(A7) $\Delta_{IC_1}(E) \wedge IC_2 \vdash \Delta_{IC_1 \wedge IC_2}(E)$

(A8) If $\Delta_{IC_1}(E) \wedge IC_2$ is consistent, then $\Delta_{IC_1 \wedge IC_2}(E) \vdash \Delta_{IC_1}(E) \wedge IC_2$

These postulates fail when this new approach to integrity constraints is adopted. Take the following example:

- $K_1 = \{w_1, w_2\}$

- $K_2 = \{w_4, w_5, w_7\}$
- $IC_1 = \neg w_5$
- $IC_2 = \neg w_4$
- $\Delta^{\text{avg},\Sigma}_{IC_1}(E) = \{w_1, w_2, w_3, w_4\}$
- $\Delta^{\text{avg},\Sigma}_{IC_1}(E) \wedge IC_2 = \{w_1, w_2, w_3\}$
- $\Delta^{\text{avg},\Sigma}_{IC_1 \wedge IC_2}(E) = \{w_1, w_3, w_5, w_7\}$

As can be seen, both A7 and A8 fail with this example: $\Delta_{IC_1}(E) \wedge IC_2 \nvdash \Delta_{IC_1 \wedge IC_2}(E)$ and $\Delta_{IC_1 \wedge IC_2}(E) \nvdash \Delta_{IC_1}(E) \wedge IC_2$.

On a final note, if we do choose to condition upon the integrity constraints before merging, this does not guarantee that the integrity constraints will not appear in the final result. Suppose we have the following:

- $IC = \neg w_1$
- $K_1 = \{w_1, w_2\}$
- $K_2 = \{w_1, w_3\}$
- $K_3 = \{w_1, w_5\}$
- After removing $w_1$ due to the integrity constraint, we get:
  - $K_{1'} = \{w_2\}$
  - $K_{2'} = \{w_3\}$
  - $K_{3'} = \{w_5\}$
- However, merging these with $\Delta^{\text{avg},\Sigma}$ still results in $\{w_1\}$. Therefore, we still need to find the results with minimal distance that satisfy the integrity constraints. In this case the final result would be $\{w_2, w_3, w_5\}$.

## References


[1] Simon D'Alfonso. Belief merging with the aim of truthlikeness. *Synthese*, pages 1–22, forthcoming. `http://theinformationalturn.net/wp-content/uploads/2015/11/merge_tr.pdf`.

[2] Sébastien Konieczny and Ramón Pino Pérez. Merging information under constraints: A logical framework. *Journal of Logic and Computation*, 12(5):773–808, 2002.